

INTRODUCTION TO THE SECOND EDITION

THE Principles of Mathematics " was published in 1903, and most of it was written in 1900. In the subsequent years the subjects of which it treats have been widely discussed, and the technique of mathematical logic has been greatly improved ; while some new problems have arisen, some old ones have been solved, and others, though they remain in a controversial condition, have taken on completely new forms. In these circumstances, it seemed useless to attempt to amend this or that, in the book, which no longer expresses my present views. Such interest as the book now possesses is historical, and consists in the fact that it represents a certain stage in the development of its subject. I have therefore altered nothing, but shall endeavour, in this Introduction, to say in what respects I adhere to the opinions which it expresses, and in what other respects subsequent research seems to me to have shown them to be erroneous.

The fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never since seen any reason to modify. This thesis was, at first, unpopular, because logic is traditionally associated with philosophy and Aristotle, so that mathematicians felt it to be none of their business, and those who considered themselves logicians resented being asked to master a new and rather difficult mathematical technique. But such feelings would have had no lasting influence if they had been unable to find support in more serious reasons for doubt. These reasons are, broadly speaking, of two opposite kinds : first, that there are certain unsolved difficulties in mathematical logic, which make it appear less certain than mathematics is believed to be ; and secondly that, if the logical basis of mathematics is accepted, it justifies, or tends to justify, much work, such as that of Georg Cantor, which is viewed with suspicion by many mathematicians on account of the unsolved paradoxes which it shares with logic. These two opposite lines of criticism are represented by the formalists, led by Hilbert, and the intuitionists, led by Brouwer.

The formalist interpretation of mathematics is by no means new, but for our purposes we may ignore its older forms. As presented by Hilbert, for example in the sphere of number, it consists in leaving the integers undefined, but asserting concerning them such axioms as shall make

possible the deduction of the usual arithmetical propositions. That is to say, we do not assign any meaning to our symbols 0, 1, 2, . . . except that they are to have certain properties enumerated in the axioms. These symbols are, therefore, to be regarded as variables. The later integers may be defined when 0 is given, but 0 is to be merely something having the assigned characteristics. Accordingly the symbols 0, 1, 2, . . . do not represent one definite series, but any progression whatever. The formalists have forgotten that numbers are needed, not only for doing sums, but for counting. Such propositions as "There were 12 Apostles" or "London has 6,000,000 inhabitants" cannot be interpreted in their system. For the symbol "0" may be taken to mean any finite integer, without thereby making any of Hilbert's axioms false; and thus every number-symbol becomes infinitely ambiguous. The formalists are like a watchmaker who is so absorbed in making his watches look pretty that he has forgotten their purpose of telling the time, and has therefore omitted to insert any works.

There is another difficulty in the formalist position, and that is as regards existence. Hilbert assumes that if a set of axioms does not lead to a contradiction, there must be some set of objects which satisfies the axioms; accordingly, in place of seeking to establish existence theorems by producing an instance, he devotes himself to methods of proving the self-consistency of his axioms. For him, "existence," as usually understood, is an unnecessarily metaphysical concept, which should be replaced by the precise concept of non-contradiction. Here, again, he has forgotten that arithmetic has practical uses. There is no limit to the systems of non-contradictory axioms that might be invented. Our reasons for being specially interested in the axioms that lead to ordinary arithmetic lie outside arithmetic, and have to do with the application of number to empirical material. This application itself forms no part of either logic or arithmetic; but a theory which makes it *a priori* impossible cannot be right. The logical definition of numbers makes their connection with the actual world of countable objects intelligible; the formalist theory does not.

The intuitionist theory, represented first by Brouwer and later by Weyl, is a more serious matter. There is a philosophy associated with the theory, which, for our purposes, we may ignore; it is only its bearing on logic and mathematics that concerns us. The essential point here is the refusal to regard a proposition as either true or false unless some method exists of deciding the alternative. Brouwer denies the law of excluded middle where no such method exists. This destroys, for example, the proof that there are more real numbers than rational numbers, and that, in the series of real numbers, every progression has a limit. Consequently large parts of analysis, which for centuries have been thought well established, are rendered doubtful.

Associated with this theory is the doctrine called finitism, which calls in question propositions involving infinite collections or infinite series, on the ground that such propositions are unverifiable. This

doctrine is an aspect of thorough-going empiricism, and must, if taken seriously, have consequences even more destructive than those that are recognized by its advocates. Men, for example, though they form a finite class, are, practically and empirically, just as impossible to enumerate as if their number were infinite. If the finitist's principle is admitted, we must not make *any* general statement—such as “All men are mortal”—about a collection defined by its properties, not by actual mention of all its members. This would make a clean sweep of all science and of all mathematics, not only of the parts which the intuitionists consider questionable. Disastrous consequences, however, cannot be regarded as proving that a doctrine is false; and the finitist doctrine, if it is to be disproved, can only be met by a complete theory of knowledge. I do not believe it to be true, but I think no short and easy refutation of it is possible.

An excellent and very full discussion of the question whether mathematics and logic are identical will be found in Vol. III. of Jørgensen's “Treatise of Formal Logic,” pp. 57-200, where the reader will find a dispassionate examination of the arguments that have been adduced against this thesis, with a conclusion which is, broadly speaking, the same as mine, namely that, while quite new grounds have been given in recent years for refusing to reduce mathematics to logic, none of these grounds is in any degree conclusive.

This brings me to the definition of mathematics which forms the first sentence of the “Principles.” In this definition various changes are necessary. To begin with, the form “ p implies q ” is only one of many logical forms that mathematical propositions may take. I was originally led to emphasise this form by the consideration of Geometry. It was clear that Euclidean and non-Euclidean systems alike must be included in pure mathematics, and must not be regarded as mutually inconsistent; we must, therefore, only assert that the axioms imply the propositions, not that the axioms are true and therefore the propositions are true. Such instances led me to lay undue stress on implication, which is only one among truth-functions, and no more important than the others. Next: when it is said that “ p and q are propositions containing one or more variables,” it would, of course, be more correct to say that they are propositional functions; what is said, however, may be excused on the ground that propositional functions had not yet been defined, and were not yet familiar to logicians or mathematicians.

I come next to a more serious matter, namely the statement that “neither p nor q contains any constants except logical constants.” I postpone, for the moment, the discussion as to what logical constants are. Assuming this known, my present point is that the absence of non-logical constants, though a necessary condition for the mathematical character of a proposition, is not a sufficient condition. Of this, perhaps, the best examples are statements concerning the number of things in the world. Take, say: “There are at least three things in the world.” This is equivalent to: “There exist objects x, y, z , and properties φ, ψ, χ , such that

x but not y has the property φ , x but not z has the property ψ , and y but not z has the property χ ." This statement can be enunciated in purely logical terms, and it can be logically proved to be true of classes of classes of classes : of these there must, in fact, be at least 4, even if the universe did not exist. For in that case there would be one class, the null-class ; two classes of classes, namely, the class of no classes and the class whose only member is the null class ; and four classes of classes of classes, namely the one which is null, the one whose only member is the null class of classes, the one whose only member is the class whose only member is the null class, and the one which is the sum of the two last. But in the lower types, that of individuals, that of classes, and that of classes of classes, we cannot logically prove that there are at least three members. From the very nature of logic, something of this sort is to be expected ; for logic aims at independence of empirical fact, and the existence of the universe is an empirical fact. It is true that if the world did not exist, logic-books would not exist ; but the existence of logic-books is not one of the premisses of logic, nor can it be inferred from any proposition that has a right to be in a logic-book.

In practice, a great deal of mathematics is possible without assuming the existence of anything. All the elementary arithmetic of finite integers and rational fractions can be constructed ; but whatever involves infinite classes of integers becomes impossible. This excludes real numbers and the whole of analysis. To include them, we need the "axiom of infinity," which states that, if n is any finite number, there is at least one class having n members. At the time when I wrote the "Principles," I supposed that this could be proved, but by the time that Dr. Whitehead and I published "Principia Mathematica," we had become convinced that the supposed proof was fallacious.

The above argument depends upon the doctrine of types, which, although it occurs in a crude form in Appendix B of the "Principles," had not yet reached the stage of development at which it showed that the existence of infinite classes cannot be demonstrated logically. What is said as to existence-theorems in the last paragraph of the last chapter of the "Principles" (pp. 497-8) no longer appears to me to be valid : such existence-theorems, with certain exceptions, are, I should now say, examples of propositions which can be *enunciated* in logical terms, but can only be proved or disproved by empirical evidence.

Another example is the multiplicative axiom, or its equivalent, Zermelo's axiom of selection. This asserts that, given a set of mutually exclusive classes, none of which is null, there is at least one class consisting of one representative from each class of the set. Whether this is true or not, no one knows. It is easy to imagine universes in which it would be true, and it is impossible to prove that there are possible universes in which it would be false ; but it is also impossible (at least, so I believe) to prove that there are no possible universes in which it would be false. I did not become aware of the necessity for this axiom until a year after the "Principles" was published. This book contains, in consequence, certain errors, for example the assertion, in §119 (p. 123), that the two definitions

of infinity are equivalent, which can only be proved if the multiplicative axiom is assumed.

Such examples—which might be multiplied indefinitely—show that a proposition may satisfy the definition with which the “Principles” opens, and yet may be incapable of logical or mathematical proof or disproof. All mathematical propositions are included under the definition (with certain minor emendations), but not all propositions that are included are mathematical. In order that a proposition may belong to mathematics it must have a further property: according to some it must be “tautological,” and according to Carnap it must be “analytic.” It is by no means easy to get an exact definition of this characteristic; moreover, Carnap has shown that it is necessary to distinguish between “analytic” and “demonstrable,” the latter being a somewhat narrower concept. And the question whether a proposition is or is not “analytic,” or “demonstrable” depends upon the apparatus of premisses with which we begin. Unless, therefore, we have some criterion as to admissible logical premisses, the whole question as to what are logical propositions becomes to a very considerable extent arbitrary. This is a very unsatisfactory conclusion, and I do not accept it as final. But before anything more can be said on this subject, it is necessary to discuss the question of “logical constants,” which play an essential part in the definition of mathematics in the first sentence of the “Principles.”

There are three questions in regard to logical constants: First, are there such things? Second, how are they defined? Third, do they occur in the propositions of logic? Of these questions, the first and third are highly ambiguous, but their various meanings can be made clearer by a little discussion.

First: Are there logical constants? There is one sense of this question in which we can give a perfectly definite affirmative answer: in the linguistic or symbolic expression of logical propositions, there are words or symbols which play a constant part, *i.e.*, make the same contribution to the significance of propositions wherever they occur. Such are, for example, “or,” “and,” “not,” “if-then,” “the null-class,” “0,” “1,” “2,” . . . The difficulty is that, when we analyse the propositions in the written expression of which such symbols occur, we find that they have no constituents corresponding to the expressions in question. In some cases this is fairly obvious: not even the most ardent Platonist would suppose that the perfect “or” is laid up in heaven, and that the “or’s” here on earth are imperfect copies of the celestial archetype. But in the case of numbers this is far less obvious. The doctrines of Pythagoras, which began with arithmetical mysticism, influenced all subsequent philosophy and mathematics more profoundly than is generally realized. Numbers were immutable and eternal, like the heavenly bodies; numbers were intelligible: the science of numbers was the key to the universe. The last of these beliefs has misled mathematicians and the Board of Education down to the present day. Consequently, to say that numbers are symbols which mean nothing appears as a horrible form of atheism. At the time

when I wrote the "Principles," I shared with Frege a belief in the Platonic reality of numbers, which, in my imagination, peopled the timeless realm of Being. It was a comforting faith, which I later abandoned with regret. Something must now be said of the steps by which I was led to abandon it.

In Chapter IV of the "Principles" it is said that "every word occurring in a sentence must have *some* meaning"; and again "Whatever may be an object of thought, or may occur in any true or false proposition, or can be counted as *one*, I call a *term*. . . . A man, a moment, a number, a class, a relation, a chimæra, or anything else that can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false." This way of understanding language turned out to be mistaken. That a word "must have *some* meaning"—the word, of course, being not gibberish, but one which has an intelligible use—is not always true if taken as applying to the word in isolation. What is true is that the word contributes to the meaning of the sentence in which it occurs; but that is a very different matter.

The first step in the process was the theory of descriptions. According to this theory, in the proposition "Scott is the author of Waverley," there is no constituent corresponding to "the author of Waverley": the analysis of the proposition is, roughly: "Scott wrote Waverley, and whoever wrote Waverley was Scott"; or, more accurately: "The propositional function '*x wrote Waverley*' is equivalent to *x is Scott*' is true for all values of *x*." This theory swept away the contention—advanced, for instance, by Meinong—that there must, in the realm of Being, be such objects as the golden mountain and the round square, since we can talk about them. "The round square does not exist" had always been a difficult proposition; for it was natural to ask "What is it that does not exist?" and any possible answer had seemed to imply that, in some sense, there is such an object as the round square, though this object has the odd property of not existing. The theory of descriptions avoided this and other difficulties.

The next step was the abolition of classes. This step was taken in "Principia Mathematica," where it is said: "The symbols for classes, like those for descriptions, are, in our system, incomplete symbols; their *uses* are defined, but they themselves are not assumed to mean anything at all. . . . Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects" (Vol. I, pp. 71-2). Seeing that cardinal numbers had been defined as classes of classes, they also became "merely symbolic or linguistic conveniences." Thus, for example, the proposition " $1 + 1 = 2$," somewhat simplified, becomes the following: "Form the propositional function '*a is not b*, and whatever *x* may be, *x is a* γ is always equivalent to *x is a or x is b*'; form also the propositional function '*a is a* γ , and, whatever *x* may be, *x is a* γ but is not *a* is always equivalent to *x is b*.'" Then, whatever γ may be, the assertion that one of these propositional functions is not always false (for different values of *a* and *b*) is equivalent to the assertion that the other is not always false." Here the numbers 1 and 2 have entirely disappeared. and a similar analysis can be applied to any arithmetical proposition.

Dr. Whitehead, at this stage, persuaded me to abandon points of space, instants of time, and particles of matter, substituting for them logical constructions composed of events. In the end, it seemed to result that none of the raw material of the world has smooth logical properties, but that whatever appears to have such properties is constructed artificially in order to have them. I do not mean that statements apparently about points or instants or numbers, or any of the other entities which Occam's razor abolishes, are false, but only that they need interpretation which shows that their linguistic form is misleading, and that, when they are rightly analysed, the pseudo-entities in question are found to be not mentioned in them. "Time consists of instants," for example, may or may not be a true statement, but in either case it mentions neither time nor instants. It may, roughly, be interpreted as follows: Given any event x , let us define as its "contemporaries" those which end after it begins, but begin before it ends; and among these let us define as "initial contemporaries" of x those which are not wholly later than any other contemporaries of x . Then the statement "time consists of instants" is true if, given any event x , every event which is wholly later than some contemporary of x is wholly later than some initial contemporary of x . A similar process of interpretation is necessary in regard to most, if not all, purely logical constants.

Thus the question whether logical constants occur in the propositions of logic becomes more difficult than it seemed at first sight. It is, in fact, a question to which, as things stand, no definite answer can be given, because there is no exact definition of "occurring in" a proposition. But something can be said. In the first place, no proposition of logic can mention any particular object. The statement "If Socrates is a man and all men are mortal, then Socrates is mortal" is not a proposition of logic; the logical proposition of which the above is a particular case is: "If x has the property of φ , and whatever has the property φ has the property ψ , then x has the property ψ , whatever x , φ , ψ may be." The word "property," which occurs here, disappears from the correct symbolic statement of the proposition; but "if-then," or something serving the same purpose, remains. After the utmost efforts to reduce the number of undefined elements in the logical calculus, we shall find ourselves left with two (at least) which seem indispensable: one is incompatibility; the other is the truth of all values of a propositional function. (By the "incompatibility" of two propositions is meant that they are not both true.) Neither of these looks very substantial. What was said earlier about "or" applies equally to incompatibility; and it would seem absurd to say that generality is a constituent of a general proposition.

Logical constants, therefore, if we are to be able to say anything definite about them, must be treated as part of the language, not as part of what the language speaks about. In this way, logic becomes much more linguistic than I believed it to be at the time when I wrote the "Principles." It will still be true that no constants except logical

constants occur in the verbal or symbolic expression of logical propositions, but it will not be true that these logical constants are names of objects, as "Socrates" is intended to be.

To define logic, or mathematics, is therefore by no means easy except in relation to some given set of premisses. A logical premiss must have *certain* characteristics which can be defined: it must have complete generality, in the sense that it mentions no particular thing or quality; and it must be true in virtue of its form. Given a definite set of logical premisses, we can define logic, *in relation to them*, as whatever they enable us to demonstrate. But (1) it is hard to say what makes a proposition true in virtue of its form; (2) it is difficult to see any way of proving that the system resulting from a given set of premisses is complete, in the sense of embracing everything that we should wish to include among logical propositions. As regards this second point, it has been customary to accept current logic and mathematics as a datum, and seek the fewest premisses from which this datum can be reconstructed. But when doubts arise—as they have arisen—concerning the validity of certain parts of mathematics, this method leaves us in the lurch.

It seems clear that there must be some way of defining logic otherwise than in relation to a particular logical language. The fundamental characteristic of logic, obviously, is that which is indicated when we say that logical propositions are true in virtue of their form. The question of demonstrability cannot enter in, since every proposition which, in one system, is deduced from the premisses, might, in another system, be itself taken as a premiss. If the proposition is complicated, this is inconvenient, but it cannot be impossible. All the propositions that are demonstrable in any admissible logical system must share with the premisses the property of being true in virtue of their form; and all propositions which are true in virtue of their form ought to be included in any adequate logic. Some writers, for example Carnap in his "Logical Syntax of Language," treat the whole problem as being more a matter of linguistic choice than I can believe it to be. In the above-mentioned work, Carnap has two logical languages, one of which admits the multiplicative axiom and the axiom of infinity, while the other does not. I cannot myself regard such a matter as one to be decided by our arbitrary choice. It seems to me that these axioms either do, or do not, have the characteristic of formal truth which characterizes logic, and that in the former event every logic must include them, while in the latter every logic must exclude them. I confess, however, that I am unable to give any clear account of what is meant by saying that a proposition is "true in virtue of its form." But this phrase, inadequate as it is, points, I think, to the problem which must be solved if an adequate definition of logic is to be found.

I come finally to the question of the contradictions and the doctrine of types. Henri Poincaré, who considered mathematical logic to be no help in discovery, and therefore sterile, rejoiced in the contradictions: "La logistique n'est plus stérile; elle engendre la contradiction!"

All that mathematical logic did, however, was to make it evident that contradictions follow from premisses previously accepted by all logicians, however innocent of mathematics. Nor were the contradictions all new ; some dated from Greek times.

In the "Principles," only three contradictions are mentioned : Burali Forti's concerning the greatest ordinal, the contradiction concerning the greatest cardinal, and mine concerning the classes that are not members of themselves (pp. 323, 366, and 101). What is said as to possible solutions may be ignored, except Appendix B, on the theory of types ; and this itself is only a rough sketch. The literature on the contradictions is vast, and the subject is still controversial. The most complete treatment of the subject known to me is to be found in Carnap's "Logical Syntax of Language" (Kegan Paul, 1937). What he says on the subject seems to me either right or so difficult to refute that a refutation could not possibly be attempted in a short space. I shall, therefore, confine myself to a few general remarks.

At first sight, the contradictions seem to be of three sorts : those that are mathematical, those that are logical, and those that may be suspected of being due to some more or less trivial linguistic trick. Of the definitely mathematical contradictions, those concerning the greatest ordinal and the greatest cardinal may be taken as typical.

The first of these, Burali Forti's, is as follows : Let us arrange all ordinal numbers in order of magnitude ; then the last of these, which we will call N , is the greatest of ordinals. But the number of all ordinals from 0 up to N is $N+1$, which is greater than N . We cannot escape by suggesting that the series of ordinal numbers has no last term ; for in that case equally this series itself has an ordinal number greater than any term of the series, *i.e.*, greater than any ordinal number.

The second contradiction, that concerning the greatest cardinal, has the merit of making peculiarly evident the need for some doctrine of types. We know from elementary arithmetic that the number of combinations of n things any number at a time is 2^n , *i.e.*, that a class of n terms has 2^n sub-classes. We can prove that this proposition remains true when n is infinite. And Cantor proved that 2^n is always greater than n . Hence there can be no greatest cardinal. Yet one would have supposed that the class containing everything would have the greatest possible number of terms. Since, however, the number of classes of things exceeds the number of things, clearly classes of things are not things. (I will explain shortly what this statement can mean.)

Of the obviously logical contradictions, one is discussed in Chapter X : in the linguistic group, the most famous, that of the liar, was invented by the Greeks. It is as follows : Suppose a man says "I am lying." If he is lying, his statement is true, and therefore he is not lying ; if he is not lying, then, when he says he is lying, he is lying. Thus either hypothesis implies its contradictory.

The logical and mathematical contradictions, as might be expected, are not really distinguishable : but the linguistic group, according to

Ramsey*, can be solved by what may be called, in a broad sense, linguistic considerations. They are distinguished from the logical group by the fact that they introduce empirical notions, such as what somebody asserts or means; and since these notions are not logical, it is possible to find solutions which depend upon other than logical considerations. This renders possible a great simplification of the theory of types, which, as it emerges from Ramsey's discussion, ceases wholly to appear unpalatable or artificial or a mere *ad hoc* hypothesis designed to avoid the contradictions.

The technical essence of the theory of types is merely this: Given a propositional function " φx " of which all values are true, there are expressions which it is not legitimate to substitute for " x ." For example: All values of "if x is a man x is a mortal" are true, and we can infer "if Socrates is a man, Socrates is a mortal"; but we cannot infer "if the law of contradiction is a man, the law of contradiction is a mortal." The theory of types declares this latter set of words to be nonsense, and gives rules as to permissible values of " x " in " φx ." In the detail there are difficulties and complications, but the general principle is merely a more precise form of one that has always been recognized. In the older conventional logic, it was customary to point out that such a form of words as "virtue is triangular" is neither true nor false, but no attempt was made to arrive at a definite set of rules for deciding whether a given series of words was or was not significant. This the theory of types achieves. Thus, for example I stated above that "classes of things are not things." This will mean: "If ' x is a member of the class α ' is a proposition, and ' φx ' is a proposition, then ' $\varphi \alpha$ ' is not a proposition, but a meaningless collection of symbols."

There are still many controversial questions in mathematical logic, which, in the above pages, I have made no attempt to solve. I have mentioned only those matters as to which, in my opinion, there has been some fairly definite advance since the time when the "Principles" was written. Broadly speaking, I still think this book is in the right where it disagrees with what had been previously held, but where it agrees with older theories it is apt to be wrong. The changes in philosophy which seem to me to be called for are partly due to the technical advances of mathematical logic in the intervening thirty-four years, which have simplified the apparatus of primitive ideas and propositions, and have swept away many apparent entities, such as classes, points, and instants. Broadly, the result is an outlook which is less Platonic, or less realist in the mediæval sense of the word. How far it is possible to go in the direction of nominalism remains, to my mind, an unsolved question, but one which, whether completely soluble or not, can only be adequately investigated by means of mathematical logic.

* Foundations of Mathematics, Kegan Paul, 1931, p.20 ff.